Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 15, 2004*
Accepted: *May 12, 2004*

$\mathcal{N} = 1^*$ model and glueball superpotential from Renormalization-Group-improved perturbation theory

Stefano Arnone, Francesco Guerrieri and Kensuke Yoshida

Dipartimento di Fisica, Università di Roma "La Sapienza"
P.le Aldo Moro, 2 – 00185 Roma, Italy and
I.N.F.N., Sezione di Roma I
E-mail: stefano.arnone@roma1.infn.it, francesco.guerrieri@roma1.infn.it, kensuke.yoshida@roma1.infn.it

Abstract: A method for computing the low-energy non-perturbative properties of SUSY GFT, starting from the microscopic lagrangian model, is presented. The method relies on covariant SUSY Feynman graph techniques, adapted to low energy, and Renormalization-Group-improved perturbation theory. We apply the method to calculate the glueball superpotential in $\mathcal{N} = 1$ SU(2) SYM and obtain a potential of the Veneziano-Yankielowicz type.

Keywords: Renormalization Group, Nonperturbative Effects, Supersymmetric Effective Theories.

JHEP05(2004)031

Contents

1. Introduction

The study of dualities has given some unexpected insight into the non-perturbative aspects of supersymmetric gauge field theories (SGFT), as it was realized that non-perturbative effects at strong coupling can often be captured by some weakly-coupled dual theory. The celebrated AdS/CFT correspondence [1] is a clear example of the power of such an approach.

Another example is the duality between matrix models (MM) and SGFT [2], that is the fact that the non-perturbative computation of the effective superpotential in some SGFT reduces to a perturbative calculation in a matrix model whose action is the tree-level superpotential.

The main idea of the MM approach to SGFT is just to integrate out the heavy "matter" fields (hypermultiplets) to get the effective glueball superpotentials, or more precisely the perturbative corrections to them.

Actually, as was shown in [3], appealing to the "duality" with the MM can be considered just as a purely technical, efficient way to calculate the relevant covariant Feynman graphs [4], corresponding to the low-energy configurations.

Conservatively [5], one can in fact compute the perturbative corrections to the leading term in the gluon superpotential — the Veneziano-Yankielowicz (VY) potential [6] — adding this latter by hand.

Recently, there have been calculations of the effective superpotential by means of analogous techniques which included the leading VY term [7, 8]. In [7], standard field theoretical manipulations with super Feynman graphs are employed, which requires the introduction of a ultraviolet cutoff, while [8] relies on MM techniques.

In the latter, one considers a $\mathcal{N}=1\,\mathrm{U}(N_c)$ supersymmetric Yang-Mills (SYM) theory with an additional chiral superfield in the adjoint representation of the gauge group and computes the equivalent MM as indicated in [2]. The lowest order, i.e. the "quadratic term" in [8], reproduces the leading VY potential of the model.

It appears that the VY potential in a pure $\mathcal{N}=1$ SYM (with no additional chiral fields) still remains to be computed. This is what we have attempted in the present note. For the sake of simplicity and clarity, we have analyzed a SU(2) SYM, the generalization to SU(N) being just a little cumbersome from a computational point of view.

Rather than make use of the MM correspondence, as in [8], we rely entirely on the covariant super Feynman graph computations described in [3], just as in [7]. However, given the perturbative equivalence of the two approaches, our results are closely related to those of [8].

In our piece of work, we have only substituted the physical massive chiral fields of [7, 8, 9] with the auxiliary fields, $(\phi_i, \bar{\phi}_i)_{i=1}^3$, in the adjoint representation of the gauge group. These latter are introduced for the purpose of regularizing the pure $\mathcal{N}=1$ SYM, following the proposal by Arkani-Hamed and Murayama [10], and can be viewed as generalized Pauli-Villars fields.

Further, instead of directly integrating out the massive fields in order to get the glueball potential, we have applied the technique of the exact renormalization group (ERG) [11, 12, 13], as adapted to our particular regularization scheme [14].

We vary the regularizing (large) mass, M_0, to a smaller M ($M \ll M_0$, eventually $M \sim 0$) while maintaining the "physics" at energy scales between M and M_0 invariant by adding suitable (generally non-local) counterterms to the original bare action. In this way, we compute the "wilsonian action", S_M, which is the solution of the well-known Polchinski's equation [11] with respect to the parameter M.

This method has the advantage of guaranteeing the absence of ultra-violet (UV) divergences in our computation and, when necessary, of supplying a "small" parameter for the systematic approximation.

Naturally, after the ERG transformation has been performed, we still have to deal with the auxiliary degrees of freedom, which are now, however, associated to the original $\mathcal{N}=4$ SYM (cf. section 4) with deforming mass $M \sim 0$. The correction they give is not expected to contribute anything to the purely holomorphic part of the superpotential [15].

In the end, we can eliminate — at least in the low-energy holomorphic sector — the auxiliary fields by the well-known limiting process $M_0 \to \infty$, $g_0 \to 0$ with the dynamically generated scale, $\Lambda_{\mathcal{N}=1}$, held fixed.

It is perhaps not too surprising that after the above limit has been taken, the "residual" superpotential for pure $\mathcal{N}=1$ SYM is exactly of the VY form with its minimum given by

$$|\bar{s}| = \left|\left\langle \frac{1}{32\pi^2} W^2 \right\rangle\right| \sim \frac{3}{\mathrm{e}} \Lambda_{N=1}^3 \,,$$

a N_c-fold solution.

JHEP05(2004)031

We will leave it to a future publication to make this value more precise and, in particular, we refrain here from commenting on the famous controversy between "weak instanton" and "strong instanton" results [16].

The paper is organized as follows. In section 2 we review the aspects of the covariant super Feynman graph calculus that will be used in the following. Then, in section 3, both our regularization scheme and the ERG transformation are outlined. The former relies on the finiteness of the mass-deformed $\mathcal{N} = 4$ SYM [17], while the latter is an adaptation of the ERG method for the standard momentum-cutoff-regularized quantum field theory [11, 12, 13]. Section 4 is devoted to describing in detail the actual calculation of the glueball superpotential. Finally, in section 5 we draw some conclusions.

2. SUSY calculus for low-energy physics

In the study of non-perturbative properties of SGFT it is possible to obtain interesting and exact results, such as the evaluation of the superpotential, by concentrating on the low-energy, holomorphic, aspects of the theory. In such a "limited" domain of application, one is led to expect that some basic QFT techniques (e.g. Feynman graphs) can be adapted and formulated in such a way that one may simplify the computation enough to be able to study some quantities of interest.

Such a technique has been put forward in [3] and applied to the "perturbative" proof of MM-SUSY QFT correspondence. It consists precisely in the modification of covariant SUSY Feynman graph techniques of Grisaru and Zanon [4], adapted to study low-energy physics. For the present work the technique allows us to partially replace the reliance on Konishi anomaly (which was needed, for instance, in [10, 14]) with a more flexible, and sometimes more precise, method of computation. In this section, we will give a brief introduction to the method of [3]. Further details can be found in [5], beside the original work. We will follow the conventions introduced in [18].

The example we have chosen is the evaluation of the holomorphic part of the partition function for chiral fields (in the adjoint representation) in an external gauge field background.

$$\mathcal{Z}(\mu,\bar{\mu}) = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi\mathcal{D}\bar{\phi} \exp i \left[\zeta \int d^4x d^4\theta \bar{\phi}\, e^V \phi + \frac{\mu}{2}\int d^2\theta \phi^2 + \frac{\bar{\mu}}{2}\int d^2\bar{\theta}\bar{\phi}^2\right], \qquad (2.1)$$

where ($\bar{\phi}$) ϕ is a (anti-)chiral superfield in the adjoint representation of the gauge group G=SU(N), and the $1/\mathcal{N}$ takes care of the trivial UV divergences (one may choose for example $\mathcal{N} = \mathcal{Z}(\mu_0, \bar{\mu}_0)$ for some appropriate μ_0, $\bar{\mu}_0$).

The first step to evaluate eq. (2.1) is to integrate out the antichiral field. We do this by going over to the gauge chiral representation

$$\hat{\phi} = \bar{\phi}\, e^V = e^{-V}\bar{\phi}\,, \qquad (2.2)$$

where the last equality holds as $\bar{\phi}$ transforms as the adjoint representation. One has

$$\begin{aligned} \overline{\nabla}_{\dot{\alpha}}\phi &= 0\,, \\ \nabla^{\alpha}\hat{\phi} &= 0\,, \end{aligned} \qquad (2.3)$$

having defined the operators appropriate for the gauge chiral representation as

$$\begin{aligned}\nabla^\alpha &\equiv \mathrm{e}^{-V} D^\alpha \mathrm{e}^V, \\ \overline{\nabla}_{\dot\alpha} &\equiv \overline{D}_{\dot\alpha}\,.\end{aligned}$$

By making use of the generalization (covariantization) of usual relationships, it is possible to reexpress the integrals on the chiral or antichiral subspace only as integrals on the full Grassmannian space:

$$\int d^4x d^2\bar\theta\,\bar\phi^2 = \int d^4x d^2\bar\theta\,\hat\phi^2 = \int d^4x d^2\theta d^2\bar\theta\,\hat\phi\left(-\frac{\overline{\nabla}^2}{4\Box_+}\right)\hat\phi\,. \tag{2.4}$$

Now it is possible to "diagonalize" the dependence upon ϕ and $\bar\phi$ in eq. (2.1) by writing:

$$\begin{aligned}\zeta\int d^4x d^4\theta\bar\phi\,\mathrm{e}^V\phi + \frac{\bar\mu}{2}\int d^4x d^2\bar\theta\bar\phi^2 &= \int d^4x d^4\theta\Big[-\frac{\bar\mu}{2}\left(\hat\phi - \frac{\zeta}{4\bar\mu}\nabla^2\phi\right)\frac{\overline{\nabla}^2}{4\Box_+}\left(\hat\phi - \frac{\zeta}{4\bar\mu}\nabla^2\phi\right) + \\ &\qquad + \frac{\zeta^2}{32\bar\mu}\nabla^2\phi\frac{\overline{\nabla}^2}{4\Box_+}\nabla^2\phi\Big] \\ &= \frac{\bar\mu}{2}\int d^4x d^2\bar\theta\left(\hat\phi - \frac{\zeta}{4\bar\mu}\nabla^2\phi\right)^2 + \frac{\zeta^2}{2\bar\mu}\int d^4x d^2\theta\phi\Box_+\phi,\end{aligned} \tag{2.5}$$

where we have used the conventions:

$$\begin{aligned}W_\alpha &= -\frac{1}{4}\bar D^2\mathrm{e}^V D_\alpha\,\mathrm{e}^{-V}, \\ \Box_{\rm cov} &= -\frac{1}{2}\{\nabla_\alpha,\overline{\nabla}_{\dot\alpha}\}\{\nabla^\alpha,\overline{\nabla}^{\dot\alpha}\}, \\ \Box_+ &= \Box_{\rm cov} - W_\alpha\nabla^\alpha - \frac{1}{2}\nabla^\alpha W_\alpha.\end{aligned} \tag{2.6}$$

So far our transformations of eq. (2.1) are exact, being only algebraic manipulations. Now we introduce a series of simplifications valid only for computations of low-energy physics, such as the determination of the superpotential. In particular, following [3], we assume that

(i) $S \equiv W^2$ can be treated as a constant; this of course implies that W^α is covariantly constant, e.g. $\{\nabla_\alpha,\overline{\nabla}_{\dot\alpha}\}W^\beta = 0$;

(ii) the term $\nabla^\alpha W_\alpha$ in eq. (2.6) is irrelevant;

(iii) moreover, moving to the new gauge:

$$\phi \to \phi' = \mathrm{e}^V\phi, \qquad W^\alpha \to W'^\alpha = \mathrm{e}^V W^\alpha \mathrm{e}^{-V},$$

one has $\phi(\Box_{\rm cov} - W_\alpha\nabla^\alpha)\phi \Rightarrow \phi'(\Box_{\rm cov} - W'_\alpha D^\alpha)\phi'$;

(iv) $\Box_{\rm cov}$ can be replaced by the ordinary d'alembertian, $\Box$.

Under these simplifying assumptions, the partition function eq. (2.1) is reduced to

$$\mathcal{Z}(\mu,\bar\mu) \propto \int \mathcal{D}\phi \exp i \int d^4x d^2\theta \left(\frac{\zeta^2}{2\bar\mu} \phi(\Box - W_\alpha D^\alpha)\phi + \frac{\mu}{2}\phi^2 \right). \tag{2.7}$$

As is well known, the computation is much easier in momentum space. Following the lead in [3], we will Fourier-transform not only the ordinary space-time coordinates, but also the Grassmannian ones, $\theta, \bar\theta$:

$$\begin{aligned} \partial_\mu &\to -ip_\mu\,, \\ D_\alpha &\to -i\pi_\alpha\,, \end{aligned}$$

which will bring about a number of important simplifications. Eq. (2.7) can therefore be rewritten as

$$\mathcal{Z}(\mu,\bar\mu) \propto \int \mathcal{D}\phi^*(p',\pi') \exp \frac{i}{2} \int d^4p\; d^2\pi\; \phi^*(p,\pi) \left(\frac{\zeta^2}{\bar\mu}(-p^2 + i\pi^\alpha W_\alpha) + \mu \right) \phi^*(-p,-\pi)\,, \tag{2.8}$$

where ϕ^* is the Fourier transform of ϕ.

The "Feynman rules" represented in eq. (2.8) exhibit a couple of very important characteristics which can be used to simplify more complex computations.

The first is scale invariance: under the rescaling

$$p_\mu \to \sqrt{\lambda} p_\mu\,, \qquad \pi_{1,2} \to \lambda \pi_{1,2}$$

one has

$$d^4p \to \lambda^2 d^4p\,, \qquad d^2\pi \to \frac{1}{\lambda^2} d^2\pi$$

i.e. the measure $d^4p\, d^2\pi$ is scale invariant.

As a result, one can see that the value of the coefficient $\zeta^2/\bar\mu$ multiplying the momentum part in eq. (2.8) is irrelevant. If one chooses, for instance, $\zeta^2/\bar\mu = 1$, then[1]

$$\begin{aligned} \mathcal{Z}(\mu,\bar\mu) \Rightarrow Z(\mu) \propto & \int \mathcal{D}\phi^*(p',\pi')\; \exp i \int d^4p\; d^2\pi\; \frac{1}{2}\phi^*(p,\pi)(-p^2 + i\pi_\alpha W^\alpha + \mu)\phi^*(-p,-\pi) \\ &= \exp -\frac{i}{2} \int d^4p\; d^2\pi\; \mathrm{tr} \ln(p^2 + i\pi_\alpha W^\alpha + \mu), \end{aligned} \tag{2.9}$$

where in the second line we have Wick-rotated the momentum p to the euclidean one.[2]

The second concerns the π dependence. The ("one-loop") integration in eq. (2.9) contains the Grassmann $\int d^2\pi = 1/2 \int d\pi_1 d\pi_2$, which must be fully absorbed by the integrand. Thus the only non vanishing contribution comes from the order-W^2 term of the same integrand. Expanding it in powers of $W\pi$ one has

$$\begin{aligned} \mathcal{Z}(\mu) &\propto \exp \frac{i}{8} \int d^4p\; d^2\pi\; \mathrm{tr} \left\{ (p^2+\mu)^{-1}(W\pi)(p^2+\mu)^{-1}(W\pi) \right\} \\ &= \exp -\frac{i}{8}\frac{t_2(A)}{16\pi^2} \int W^2 \int_0^\infty d\tau\, \frac{\tau}{(\tau+\mu)^2}\,, \end{aligned} \tag{2.10}$$

[1] Note that this implies giving up the information on the antiholomorphic part of the action.

[2] The momentum has been rescaled to rid us of the dependence upon $\bar\mu$. Therefore the dimension of p^2 is now [mass], i.e. it is homogeneous with μ.

where $t_2(A)$ is the Dynkin index of the adjoint representation of the gauge group ($t_2(A) = N_c$ for $\mathrm{SU}(N_c)$) and $\tau = p^2$.

The above expression is divergent. On the other hand, one can compute

$$\begin{aligned}\mu\,\partial_\mu \ln \mathcal{Z}(\mu) &= \left\langle i\frac{\mu}{2}\int \phi^2\right\rangle \\ &= \frac{i}{8}\frac{t_2(A)}{16\pi^2}\int W^2 \int_0^\infty d\tau\,\frac{-2\tau\mu}{(\tau+\mu)^3} \\ &= -\frac{i}{16}\frac{t_2(A)}{8\pi^2}\int W^2\,. \end{aligned} \tag{2.11}$$

Integrating eq. (2.11) one obtains

$$\frac{\mathcal{Z}(\mu_2)}{\mathcal{Z}(\mu_1)} = \exp -\frac{i}{16}\frac{t_2(A)}{8\pi^2}\int \ln\left(\frac{\mu_2}{\mu_1}\right) W^2\,. \tag{2.12}$$

The expression eq. (2.12) shows its close connection with the Konishi anomaly [19]. As a matter of fact, if one rescales the ϕ fields in eq. (2.9) by λ, i.e. letting $\phi^*(p,\pi)\to\lambda\phi^*(p,\pi)$ and takes into account the (possibly non-trivial) corresponding jacobian,

$$\int \mathcal{D}\lambda\,\phi^* = J(\lambda)\int \mathcal{D}\phi^*\,, \tag{2.13}$$

then

$$\mathcal{Z}(\mu) = J(\lambda)\int \mathcal{D}\phi^*\,\exp i\int d\!\!{}^-{}^4p\, d^2\pi\frac{1}{2}\phi^*\Big[\lambda^2(-p^2+i\pi W)+\lambda^2\mu\Big]\phi^*. \tag{2.14}$$

Now, exploiting the scale invariance of the measure $d\!\!{}^-{}^4p\;d^2\pi$, the λ^2 multiplying the momentum part can be set to one, leaving

$$\mathcal{Z}(\mu) = J(\lambda)\int \mathcal{D}\phi^*\,\exp i\int d\!\!{}^-{}^4p\, d^2\pi\frac{1}{2}\phi^*\Big[-p^2+i\pi W+\lambda^2\mu\Big]\phi^* = J(\lambda)\mathcal{Z}(\lambda^2\mu)\,. \tag{2.15}$$

Thus $J(\lambda) = \mathcal{Z}(\mu)/\mathcal{Z}(\lambda^2\mu) = \exp\frac{i}{8}\frac{t_2(A)}{8\pi^2}\int(\ln\lambda)W^2$, which is indeed the correct value [19].

The computation illustrated above is equivalent to the old-fashioned Feynman graph method of determining the anomalies in SYM model [20, 14]. The method proposed in [3], instead, recognizes the common principles in those approximate computations and reformulates them as a method for efficiently extracting the holomorphic part of the superpotential.

3. Regularized $\mathcal{N}=1$ model

3.1 $\mathcal{N}=1^*$ model

In the absence of a general, symmetry-preserving cut-off scheme for supersymmetric gauge field theories [21], we adopt the following regularization, applicable only to the limited class of SGFT models originally suggested by Arkani-Hamed and Murayama [10].

JHEP05(2004)031

We make use of the fact that four dimensional $\mathcal{N}=4$ SYM is UV finite [22, 17]. The classical action for the model is

$$S_{\mathcal{N}=4}(V,\phi_i,\bar{\phi}_i;g_0) = \frac{1}{16}\int d^4x\,d^2\theta\frac{1}{\hat{g}_0^2}W^a_\alpha W^{a\alpha} + \int dx\,d^4\theta\left(\frac{2}{g_0^2}\right)t_2(A)\times$$
$$\times\sum_{i=1}^{3}\bar{\phi}_i e^V\phi_i + \int d^4x\,d^2\theta\left(\frac{\sqrt{2}}{2g_0^2}\right)if_{abc}\,\phi^i_a\,\phi^j_b\,\phi^k_c\,\frac{\epsilon^{ijk}}{3!} + \text{h.c.}, \qquad (3.1)$$

all the relevant fields transforming as the adjoint representation of the gauge group. Eq. (3.1) is written in the so-called holomorphic form, which is equivalent to the more usual canonical form with the rescaling $W^a(V)\to W^a(g_cV_c)$. (However one must pay attention to the corresponding Konishi anomaly [19, 10]. See appendix A.) The holomorphic gauge coupling constant, $\hat{g}_0$, is given by

$$\frac{1}{\hat{g}_0^2} = \frac{1}{g_0^2} + i\frac{\theta_0}{8\pi^2}\,. \qquad (3.2)$$

Now it is known [17] that the quantization of the model with classical action $S_{\mathcal{N}=4}$ is still free from UV divergences even in the presence of mass deformations, i.e. terms of the form

$$\frac{1}{2}\sum_{i=1}^{3}\int d^2\theta\,M_{0i}\,\phi_i^2 + \text{h.c.} \qquad (3.3)$$

Moreover, it is believed to be also free of infra-red (IR) divergences if all the external states are gauge invariant [17].

Thus, we assume that the partition function

$$\mathcal{Z}_{M_0}(\text{sources}) = \int\mathcal{D}V\int\prod_i\mathcal{D}\phi_i\,\mathcal{D}\bar{\phi}_i\exp i\Bigg[S_{\mathcal{N}=4}(V,\phi_i,\bar{\phi}_i;g_0) + \frac{1}{2}\sum_{i=1}^{3}\int M_{0i}\,\phi_i^2 +$$
$$+\,\text{h.c.} + \text{gauge invariant sources}\Bigg] \qquad (3.4)$$

be well defined for an arbitrary set of masses $\vec{M}_0 = (M_{0i})^3_{i=1}$.

By choosing the special case,

$$\vec{M}_0 = (M_0, M_0, M_0)\,, \qquad (3.5)$$

one can realize the regularized model which, at energy scales much lower than M_0, gives the physics of $\mathcal{N}=1$ SYM.

In the limit that $M_0\to\infty$ and $g_0\to 0$ with

$$\Lambda_{\mathcal{N}=1} = M_0\exp -\frac{8\pi^2}{3N_c g_0^2}\,, \qquad (3.6)$$

held fixed, we have pure $\mathcal{N}=1$ SYM.

JHEP05(2004)031

3.2 Renormalization group transformation

As has been suggested in [10, 14], we can compute the effective "wilsonian" action, S_M, by varying the regularizing mass, M_0 (which is to be much bigger than any physical scale we are interested in) to a much smaller value, M, while keeping the physics (that is the numerical value of $\mathcal{Z}_{M_0}$) unchanged. We look for the transformation which implements the equivalence relation:

$$\begin{aligned} \mathcal{Z}_{M_0} &= \int \mathcal{D}V \int \mathcal{D}\phi_i\, \mathcal{D}\bar{\phi}_i \exp i \left[S_{\mathcal{N}=4}(V,\phi_i,\bar{\phi}_i;g_0) + \frac{M_0}{2}\sum_{i=1}^{3}\int \phi_i^2 + \text{h.c.} + \text{sources}\right] \\ &= \int \mathcal{D}V \int \mathcal{D}\phi_i\, \mathcal{D}\bar{\phi}_i \exp i \left[S_M(V,\phi_i,\bar{\phi}_i) + \frac{M}{2}\sum_{i=1}^{3}\int \phi_i^2 + \text{h.c.} + \text{ren. sources}\right]. \end{aligned} \tag{3.7}$$

In general, S_M is non-local and expressed as a functional integral over some auxiliary fields. This is the adaptation of the so-called Exact Renormalization Group (ERG) method for the usual momentum-cutoff-regulated quantum field theory [11, 12, 13], where the equality

$$\begin{aligned} &\int_{0<|p|<\Lambda_0} \mathcal{D}\Phi^*(p)\, \exp(-S'_{\Lambda_0}(\Phi) + \text{sources}) = \\ &\qquad = \int_{0<|p|<\Lambda} \mathcal{D}\Phi^*(p)\, \exp(-S'_{\Lambda}(\Phi) + \text{ren. sources}) \end{aligned} \tag{3.8}$$

is required. In the above, $S'_{\Lambda_0}(\Phi)$ is the bare action with UV cutoff Λ_0.

In some simple models, we can implement the condition $0 < |p| < \Lambda(\Lambda_0)$ at the level of the propagator by singling out the regulated kinetic term from the action, i.e.

$$S'_\Lambda(\Phi) = \frac{1}{2}\int \Phi^*(-p)\frac{D^{-1}(p)}{K_\Lambda(p)}\Phi^*(p) + S_\Lambda(\Phi)\,, \tag{3.9}$$

where $K_\Lambda(p)$ is the cutoff profile corresponding to $0 < |p| < \Lambda$ and again Φ^* is the Fourier-transformed field.

S_Λ satisfies the well-known Polchinski's equation [11]

$$\Lambda\partial_\Lambda S_\Lambda(\Phi) = -\frac{1}{2}\int d^4p\; D(p)\,\Lambda\partial_\Lambda K_\Lambda \left(\frac{\delta^2 S_\Lambda}{\delta\Phi^*(p)\,\delta\Phi^*(-p)} - \frac{\delta S_\Lambda}{\delta\Phi^*(p)}\frac{\delta S_\Lambda}{\delta\Phi^*(-p)} \right). \tag{3.10}$$

One can easily see, then, that in the limit that $\Lambda \to 0$, S_Λ contains all the information about the complete solution of the original model with ultraviolet cutoff Λ_0, i.e.

$$\lim_{\Lambda\to 0} S_\Lambda(\Phi) = W_{\Lambda_0}(J)\,, \tag{3.11}$$

with $W_{\Lambda_0}(J)$ being the generating functional of connected Green's functions with UV cutoff Λ_0 [13].

As has been shown in [14], our action $S_M(V,\phi_i,\bar{\phi}_i)$ in eq. (3.7) satisfies the analogue of eq. (3.10) with Λ replaced by M when the contribution of the Konishi anomaly is subtracted [14].

JHEP05(2004)031

From a physical point of view, we may assume that the analogue of eq. (3.11) is also valid for our simplified regularization and corresponding RG transformation, eq. (3.7). Therefore we assume that for small enough M ($M \ll M_0$), the action $S_M(V, \phi_i, \bar{\phi}_i)$ in eq. (3.7), i.e. the solution to the anomaly-corrected Polchinski's equation, describes with good approximation the physics at energy scales $M \leq E \ll M_0$, with no further quantization procedure (path integral).

We can compute S_M by generalising the Zinn-Justin's transformation [23, 24], originally in the form:

$$\int \mathcal{D}\Phi \exp\left[-\frac{1}{2}\int \Phi^*(-p)\frac{1}{D_1(p)+D_2(p)}\Phi^*(p) - V(\Phi)\right] =$$
$$= \int \mathcal{D}\Phi\, \mathcal{D}\Phi' \exp\Bigg[-\frac{1}{2}\int \Phi^*(-p)D_1^{-1}(p)\Phi^*(p) - \frac{1}{2}\int \Phi'^*(-p)D_2^{-1}(p)\Phi'^*(p) -$$
$$- V(\Phi+\Phi')\Bigg] \tag{3.12}$$

to the much simpler form of gaussian integration:

$$\exp i\frac{M_0}{2}\int \phi_i^2 = \frac{\int \mathcal{D}\phi_i' \,\exp i\int\left[\frac{M}{2}(\phi_i-\phi_i')^2 + \frac{\widetilde{M}}{2}\phi_i'^2\right]}{\int \mathcal{D}\phi_i' \,\exp i\frac{M+\widetilde{M}}{2}\int \phi_i'^2}\,, \tag{3.13}$$

where $\widetilde{M}$ is the "reduced" mass, defined by $1/M_0 = 1/M + 1/\widetilde{M}$.

Inserting eq. (3.13) in the first line of eq. (3.7) yields the formal expression for S_M:

$$\exp iS_M = \frac{\int \mathcal{D}\phi_i'\,\mathcal{D}\bar{\phi}_i' \exp i\left[S_{\mathcal{N}=4}(\ldots,(\phi_i',\bar{\phi}_i')_1^3; g_0) + \frac{M}{2}\sum_{i=1}^3 \int(\phi_i-\phi_i')^2 + \text{h.c.}\right]}{\int \mathcal{D}\phi_i'\,\mathcal{D}\bar{\phi}_i' \exp i\frac{M+\widetilde{M}}{2}\sum_{i=1}^3\int \phi_i'^2 + \text{h.c.}}\,. \tag{3.14}$$

4. $\mathcal{N}=1$ Super Yang-Mills and glueball superpotential

The partition function for the regularized $\mathcal{N}=1$ SYM is given by eq. (3.4) with the mass configuration, eq. (3.5).

One has to deal with all the three auxiliary fields $(\phi_i)_{i=1}^3$ which enter the bare action $S_{\mathcal{N}=4}$ with a cubic interaction term. As the original action is still quadratic in any pair of auxilary fields (cf. eq. (3.1)) [2], in order to compute the wilsonian action S_M, one can proceed in two stages. In the first stage, one integrates out two of the three massive fields — say ϕ_1, ϕ_2 — by means of the RG transformation outlined in section 2. In the second stage, one moves on to integrating out (i.e. reducing by RG procedure) the remaining chiral field, ϕ_3.[3]

In what follows, in order to ease notation, ϕ_3 ($\bar{\phi}_3$) will be replaced by ϕ ($\bar{\phi}$).

4.1 ϕ_1 and ϕ_2 reduction

One applies the process described in the previous section keeping ϕ out as if it were an external field.

[3]A similar idea has been used in [3].

Following section 2, we integrate out $\widehat{\phi}_i = \bar{\phi}_i\, e^V, i = 1, 2$. Then $\mathcal{Z}_{M_0}$ is reduced to

$$\begin{aligned}\mathcal{Z}_{M_0} = &\int \mathcal{D}V\mathcal{D}\phi\mathcal{D}\bar{\phi}\prod_{i=1}^{2}\mathcal{D}\phi_i\mathcal{D}\widehat{\phi}_i \times \\ &\times \exp i\left[\frac{1}{2}\int d\!\!{}^-{}^4p\, d^2\pi\phi^*_{ia}(p,\pi)\left(-p^2+\pi\widehat{W}+\widehat{A}+M_0\right)_{(ia,jb)}\phi^*_{jb}(-p,-\pi)+\right. \\ &\left.\quad + \frac{M_0}{2}\int\sum_{i=1}^{2}\widehat{\phi}_i^2 + \frac{1}{g_0^2}\bar{\phi}_a(e^V)_{(ab)}\phi_b + \frac{M_0}{2}\int\phi^2 + \frac{\overline{M}_0}{2}\int\widehat{\phi}^2 + (V \text{ sources})\right].\end{aligned} \tag{4.1}$$

The matrix $\widehat{A}(\phi)$ in the above corresponds to the cubic interaction in $S_{\mathcal{N}=4}$ and it is given by

$$\widehat{A}(\phi) = \frac{1}{\sqrt{2}g_0^2}(\vec{\phi}\cdot\vec{F})_{ab}\,\epsilon_{ij}, \tag{4.2}$$

with F^a being the generators of the adjoint representation of $\mathrm{SU}(N_c)$, a, b the corresponding indices and $i, j = 1, 2$. Also $\widehat{W}(\phi) = (i\vec{W}\cdot\vec{F})_{ab}\,\delta_{ij}$.

Now eq. (4.1) is gaussian in the auxiliary fields, $\phi^*_i(p,\pi)$ (which, in this approximation, are decoupled from the antiholomorphic $\hat{\phi}_i$). Hence one may be tempted to integrate out the auxiliary fields directly so as to obtain the (low-energy part of the) effective action.

However, there are two obstacles to this line of reasoning.

First of all, the integrals over the (euclidean) 4-momentum are divergent.

Secondly and more importantly, perhaps, in some computations [25] it is useful to have a small parameter through which to attempt a systematic approximation. Indeed, the parameters with dimension of mass are M_0 and ϕ. M_0 is expected to be large and ϕ, being integrated over, can be large as well. As a consequence, there is no certainty about the order of magnitude of, e.g., the ratio ϕ/M_0.

It would be much more convenient if one could introduce some definitely small parameter, M, such that $M/|\phi| \ll 1 (M \ll M_0)$. The ERG approach outlined in section 3.2 is precisely the answer to this problem, as it amounts to lowering the regularising mass without changing the physics. The price to pay for such a strategy, though, is that one does not obtain immediately the effective action, but the "wilsonian" $S_M(V, \phi, \bar{\phi}, \phi_i, \bar{\phi}_i) - S_{\mathcal{N}=4}$ (cf. eqs. (3.7), (3.14)). Holomorphic corrections due to the integration of the original $S_{\mathcal{N}=4}$ over the auxiliary fields are not expected to be significant in our case. In fact, as shown by Dorey et al. [15], these contributions are proportional to the product $M_1M_2M_3 = M^3$, and therefore vanish in the limit that $M \to 0$.

Applying the RG transformation of section 3 (i.e. $M_0 \to M \ll M_0$), eq. (4.1) takes the form

$$\begin{aligned}\mathcal{Z}_{M_0} = &\int \mathcal{D}V\mathcal{D}\phi\mathcal{D}\bar{\phi}\prod_{i=1}^{2}\mathcal{D}\phi_i\mathcal{D}\widehat{\phi}_i\prod_{i=1}^{2}\mathcal{D}\phi'_i \times \\ &\times \exp i\left[\frac{1}{2}\int d\!\!{}^-{}^4p\, d^2\pi\phi^*_{ia}\Big(-p^2+\pi\widehat{W}+\widehat{A}+M\Big)_{(ia,jb)}\phi^*_{jb}+\right.\end{aligned}$$

$$+\phi_{ia}^{*}{}'\Big\{\Big(-p^2+\pi\widehat{W}+\widehat{A}+M_0\Big)\Big(-p^2+\pi\widehat{W}+\widehat{A}+M\Big)^{-1}\Big\}_{(ia,jb)}\phi_{jb}^{*}{}'+$$

$$+\frac{\overline{M}_0}{2}\int\sum_{i=1}^{2}\widehat{\phi}_i^2+\cdots\Big], \tag{4.3}$$

where the ellipsis refers to terms depending upon $V, \phi, \bar{\phi}$ and to source terms. (The particular form of the propagators in eq. (4.3) results from diagonalising ϕ_i, ϕ_i' in eq. (3.14).)

The $(\phi_i, \widehat{\phi}_i)$ part of the path integral should reproduce the relevant part of the original bare action $S_{\mathcal{N}=4}$, with altered mass M, while the integration over $\phi_{1,2}'$ should contribute to the non-trivial part of the Wilson action $S_M - S_{\mathcal{N}=4}$.

The gaussian integral over $\phi_{1,2}'$ gives a convergent integral over the 4-momentum. The relevant part of $\mathcal{Z}_M$ depending on $\phi_{1,2}'$ becomes:

$$\exp -\frac{i}{2}\int d\!\!{}^-{}^4 p d^2\pi\,\mathrm{tr}\Big\{\ln(p^2+\pi\widehat{W}+\widehat{A}+M_0)-\ln(p^2+\pi\widehat{W}+\widehat{A}+M)\Big\},$$

and, after integration over π_1, π_2,

$$\exp -\frac{i}{8}\int d\!\!{}^-{}^4 p\,\mathrm{tr}\Bigg\{\Bigg[(p^2+M_0)^{-2}\Big(1+\frac{\widehat{A}}{p^2+M_0}\Big)^{-1}\widehat{W}_1\Big(1+\frac{\widehat{A}}{p^2+M_0}\Big)^{-1}\widehat{W}_2-(\widehat{W}_1\leftrightarrow\widehat{W}_2)\Bigg]-$$
$$-\Bigg[(p^2+M)^{-2}\Big(1+\frac{\widehat{A}}{p^2+M}\Big)^{-1}\widehat{W}_1\Big(1+\frac{\widehat{A}}{p^2+M_0}\Big)^{-1}\widehat{W}_2-$$
$$-(\widehat{W}_1\leftrightarrow\widehat{W}_2)\Bigg]\Bigg\}, \tag{4.4}$$

where the 4-momentum p_μ has been Wick-rotated.

From now on, for the sake of definiteness and simplicity, we limit ourselves to the SU(2) gauge group.[4] Then the matrices $\widehat{A}$ and $\widehat{W}$ are explicitly:

$$\widehat{A}_{(ia,jb)} = i\phi_c\,\epsilon_{cab}\,\epsilon_{ij}\,, \tag{4.5}$$
$$\widehat{W}_{\alpha(ia,jb)} = i(W_\alpha)_c\,\epsilon_{cab}\,\delta_{ij}\,, \tag{4.6}$$

with $\alpha = 1,2$; $i,j = 1,2$; $a,b,c =$ adj (SU(2)) $= 1,2,3$.

In eq. (4.4) we have to perform the integration of a function of the form:

$$\frac{1}{(p^2+\mu)^2}\,\mathrm{tr}\Bigg\{\Big(1+\frac{\widehat{A}}{p^2+\mu}\Big)^{-1}\widehat{W}_1\Big(1+\frac{\widehat{A}}{p^2+\mu}\Big)^{-1}\widehat{W}_2\Bigg\}=$$
$$=4\left(1+\frac{\phi^2}{(p^2+\mu)^2}\right)^{-2}\left\{W_1W_2-2\frac{(\phi\cdot W_1)(\phi\cdot W_2)}{(p^2+\mu)^2}+\frac{\phi^2(W_1W_2)}{(p^2+\mu)^2}\right\}, \tag{4.7}$$

where μ stands for M_0 or M. One can still choose a special direction for the "external" field $\vec{W}$:

$$\vec{W} = (0,0,W)\,.$$

[4] Refer to [25] for the generalization to the SU(N_c) case.

Then eq. (4.7) becomes:[5]

$$4W_1W_2\left(1+\frac{\phi^2}{(p^2+\mu)^2}\right)^{-2}\left[1+\frac{\phi_1^2+\phi_2^2-\phi_3^2}{(p^2+\mu)^2}\right]. \tag{4.8}$$

Unfolding the $d\!\!{}^{-}{}^4p$ integral,

$$d\!\!{}^{-}{}^4p=\frac{1}{16\pi^2}\,p^2\,dp^2=\frac{1}{16\pi^2}\,d\tau\,\tau\,,$$

the integral to be effected in eq. (4.4) takes the form

$$\begin{aligned}\frac{4W_1W_2}{16\pi^2}\int_0^\infty d\tau\,\tau\Bigg[&\left\{\frac{(\tau+M_0)^2-\phi^2}{((\tau+M_0)^2+\phi^2)^2}-\frac{(\tau+M)^2-\phi^2}{((\tau+M)^2+\phi^2)^2}\right\}+\\&+2(\phi_1^2+\phi_2^2)\left\{\frac{1}{((\tau+M_0)^2+\phi^2)^2}-\frac{1}{((\tau+M)^2+\phi^2)^2}\right\}\Bigg].\end{aligned} \tag{4.9}$$

The terms in the first pair of curly brackets give:

$$\frac{2W_1W_2}{16\pi^2}\left[\ln\left(\frac{M}{M_0}\right)^2+\ln\left(\frac{1+\phi^2/M^2}{1+\phi^2/M_0^2}\right)\right]. \tag{4.10}$$

The terms in the second line of eq. (4.9) are proportional to the projection of ϕ in the direction orthogonal to that of $\vec{W}$ and give:

$$\frac{4W_1W_2}{2\cdot 16\pi^2}\left[\frac{2(\phi_1^2+\phi_2^2)}{M^2}\left(\frac{M}{\phi}\right)^3\left(\arctan\left(\frac{\phi}{M}\right)-\frac{\phi}{M}\right)-(M\to M_0)\right]. \tag{4.11}$$

Together they give the effective potential term for ϕ, which can now be reduced.

4.2 ϕ reduction

Now we can apply the methods of sections 2,3 to the effective action we have obtained for ϕ. We limit ourselves to the gaussian approximation of the potential terms, eqs. (4.10), (4.11). Adding the terms from $(\widehat{W}_1\leftrightarrow\widehat{W}_2)$, we obtain

$$\frac{4W_1W_2}{16\pi^2}\left[\ln\left(\frac{M}{M_0}\right)^2+\left(\frac{\phi}{M}\right)^2-\frac{2}{3}\frac{\phi_1^2+\phi_2^2}{M^2}\right], \tag{4.12}$$

where the limit $M_0\to\infty$ has been anticipated and ϕ/M_0 terms have been discarded.

Thus, to reduce ϕ and $\bar{\phi}$, we have to start from the action:

$$\begin{aligned}S_{\rm eff}(\phi,\bar\phi)=&\frac{1}{g_0^2}\int\bar\phi\,\mathrm{e}^V\phi+\frac{M_0}{2}\int\phi^2+\frac{1}{16}\frac{1}{4\pi^2}\int\Big(\frac{\phi_1^2+\phi_2^2}{3M^2}+\frac{\phi_3^2}{M^2}\Big)W^2+\\&+\frac{\overline{M}_0}{2}\int\bar\phi^2+O(\phi^4\text{term})+(\text{irrelevant non holomorphic part})\,.\end{aligned} \tag{4.13}$$

[5]In the following, unless stated otherwise, $\phi_{1,2,3}$ will refer to the three colour components of the remaining chiral superfield, ϕ.

From eq. (4.13) we can compute the contribution to S_M proceeding exactly as before, i.e. first we will integrate out $\widehat{\phi} \equiv \bar{\phi}\,\mathrm{e}^V$ and then we will apply the RG transformation $\frac{1}{2}M_0\phi^2 \to \frac{1}{2}M\phi^2$. For the configuration $\vec{V} = (0,0,V)$ it is convenient to use the following linear combinations of the fields:[6]

$$\begin{aligned} (\phi)_{1,2} &\to \phi_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm i\phi_2)\,, \\ (\phi)_3 &\to (\phi)_3\,. \end{aligned} \tag{4.14}$$

This gives

$$\begin{aligned} \bar{\phi}_a(\mathrm{e}^V)_{(ab)}\phi_b &= \bar{\phi}_+\mathrm{e}^V\phi_+ + \bar{\phi}_-\mathrm{e}^{-V}\phi_- + \bar{\phi}_3\phi_3\,, \\ \phi_1^2 + \phi_2^2 &= 2\phi_+\phi_-\,. \end{aligned} \tag{4.15}$$

After integrating out $\widehat{\phi} = \bar{\phi}\,\mathrm{e}^V$, we have the effective action:[7]

$$\begin{aligned} &\frac{1}{2}\sum_{I,J}^{+,-}\phi_I^*\left(-p^2 + i\pi W\sigma_3 + \left(M_0 + \frac{S}{32\pi^2\cdot 3\cdot M^2}\right)\sigma_1\right)_{(IJ)}\phi_J^* + \\ &\qquad + \frac{1}{2}\phi_3^*\left(-p^2 + M_0 + \frac{S}{32\pi^2\cdot M^2}\right)\phi_3^* + \frac{1}{g_0^2}(\phi\cdot\phi_1\wedge\phi_2)\,, \end{aligned} \tag{4.16}$$

where we wrote $W^2 \equiv S$.

Applying the RG transformation ($M_0 \to M$), introducing the auxiliary fields $(\phi'_\pm, \phi'_3)$ and diagonalizing, eq. (4.16) is transformed to:

$$\begin{aligned} \frac{1}{2}\int d^4p d^2\pi \Bigg[&\phi'_I\left(-p^2 + i\pi W\sigma_3 + \left(M_0 + \frac{S}{32\pi^2\cdot 3\cdot M^2}\right)\sigma_1\right)_{IK} \times \\ &\times\left(-p^2 + i\pi W\sigma_3 + \left(M + \frac{S}{32\pi^2\cdot 3\cdot M^2}\right)\sigma_1\right)^{-1}_{KJ}\phi'_J + \\ &+ \phi'_3\left(-p^2 + M_0 + \frac{S}{32\pi^2\cdot 3\cdot M^2}\right)\left(-p^2 + M + \frac{S}{32\pi^2\cdot 3\cdot M^2}\right)^{-1}\phi'_3 + \\ &+ \text{linear term in } (\phi'_\pm, \phi'_3) + (\phi_\pm, \phi_3)\text{part}\Bigg]\,, \end{aligned} \tag{4.17}$$

with $I, J, K = +, -$.

The linear term comes from the last term in eq. (4.16) after the diagonalization. Discarding it for the moment, we can effect the path integral $\int \mathcal{D}\phi'_\pm \mathcal{D}\phi'_3$. After a Wick rotation, the relevant factor for Z_{M_0} comes from the bilinear term in eq. (4.17):

$$\begin{aligned} \exp -\frac{1}{2}\int d^4p d^2\pi\, \mathrm{tr}\Bigg\{&\ln\left(p^2 + i\pi W\sigma_3 + \left(M_0 + \frac{S}{32\pi^2\cdot 3\cdot M^2}\right)\sigma_1\right) - \\ &- \ln\left(p^2 + i\pi W\sigma_3 + \left(M + \frac{S}{32\pi^2\cdot 3\cdot M^2}\right)\sigma_1\right)\Bigg\}\,. \end{aligned} \tag{4.18}$$

[6] Remember that ϕ_1, ϕ_2 and ϕ_3 all refer to the gauge indices of the third chiral field $\phi (= \phi_3)$.

[7] Please note that in eq. (4.16), $\phi_{1,2}$ refer to the two massive chiral fields, and this last linear term comes from the potential term in the original$S_{\mathcal{N}=4}$.

Note that the integral over the "neutral" field ϕ_3' gives only a vanishing contribution in our approximation scheme (cf. section 3). Effecting the π_1, π_2 integral in eq. (4.18), the exponent becomes:

$$\frac{iW^2}{4}\int d\!\!{}^-{}^4p\,\mathrm{tr}\left\{(p^2+\mathcal{M}\sigma_1)^{-1}\sigma_3(p^2+\mathcal{M}\sigma_1)^{-1}\sigma_3\right\}_{\mathcal{M}=M+\frac{S}{32\pi^2\cdot 3\cdot M^2}}^{\mathcal{M}=M_0+\frac{S}{32\pi^2\cdot 3\cdot M^2}}. \tag{4.19}$$

Effecting the integral over $d\!\!{}^-{}^4p = \tau d\tau/16\pi^2$ one obtains:

$$\frac{i}{4}\frac{W^2}{16\pi^2}\ln\left(\frac{M+\frac{S}{32\pi^2\cdot 3\cdot M^2}}{M_0+\frac{S}{32\pi^2\cdot 3\cdot M^2}}\right)^2. \tag{4.20}$$

Thus the contribution eq. (4.18) to $\mathcal{Z}_{M_0}$ becomes

$$\exp\frac{i}{4}\frac{1}{16\pi^2}\int\left\{\ln\left(\frac{M}{M_0}\right)+\ln\left(\frac{1+\frac{S}{32\pi^2\cdot 3\cdot M^3}}{1+\frac{S}{32\pi^2\cdot 3\cdot M_0M^2}}\right)\right\}S\,, \tag{4.21}$$

where, as usual, $S\equiv W^2$. To this one must add

(a) the residual constant term coming from the computation in section 4.1 (cf. eq. (4.12)):

$$\exp\frac{i}{4}\frac{1}{16\pi^2}\int\ln\left(\frac{M}{M_0}\right)^2 S;$$

(b) the gauge kinematic term in $S_{N=4}$:

$$\exp\frac{i}{16}\int\left(\frac{1}{g_0^2}+\frac{i\theta_0}{8\pi^2}\right)S=\exp\frac{i}{8}\frac{1}{16\pi^2}\int\left(2\ln\left(\frac{M_0}{\Lambda_{\mathcal{N}=1}}\right)^3+i\theta_0\right)S.$$

Putting together all the terms, one obtains

$$\exp\frac{i}{8}\frac{1}{16\pi^2}\int\left[2\ln\left(\frac{M}{\Lambda_{\mathcal{N}=1}}\right)^3+2\ln\left(\frac{1+\frac{S}{32\pi^2\cdot 3\cdot M^3}}{1+\frac{S}{32\pi^2\cdot 3\cdot M_0M^2}}\right)+i\theta_0\right]S\,. \tag{4.22}$$

Now choose M so that $S^{1/3}/M\gg 1$ and consider the pure SYM limit $M_0\to\infty$, $g_0\to 0$ with $\Lambda_{\mathcal{N}=1}$ fixed:

$$\exp\frac{i}{8}\frac{1}{16\pi^2}\int\left[2\ln\left(\frac{S}{32\pi^2\cdot 3\cdot\Lambda^3}\right)S+i\theta_0 S\right], \tag{4.23}$$

which gives the effective potential

$$W_{\text{eff}}=\frac{1}{128\pi^2}\int\left[2\ln\left(\frac{S}{3\cdot\Lambda^3\cdot 32\pi^2}\right)S+i\theta_0 S\right]. \tag{4.24}$$

Eq. (4.24) is of the VY form [6].

One can study the vacuum structure of $\mathcal{N}=1$ SYM looking for the minima of W_{eff}

$$\left(\frac{\partial W_{\text{eff}}}{\partial S}\right)_{\bar{S}}=0\,. \tag{4.25}$$

Bearing in mind that θ_0 is defined only up to $2k\pi, k = 0, \pm 1, \pm 2 \ldots$, the solution to eq. (4.25) is:

$$\frac{\bar{S}}{32\pi^2} = \frac{3}{\mathrm{e}} \left(\pm \mathrm{e}^{-\frac{i\theta_0}{2}} \right) \Lambda^3 . \tag{4.26}$$

One can now conclude that

(i) $\langle W^2 \rangle \neq 0$

(ii) the vacuum is two-fold degenerate.

For SU(N_c) one expects, instead of eq. (4.24),

$$W_{\text{eff}} \propto \int \left[N_c S \ln \left(\frac{S}{3 \cdot \Lambda^3 \cdot 32\pi^2} \right) + i\theta_0 S \right], \tag{4.27}$$

which predicts a N_c-fold degenerate vacuum.

4.3 Linear term

In eq. (4.17) we have left out the linear term in the auxiliary fields $\phi'_\pm$ and ϕ'_3. The effect of such a term is to generate an additional four-point vertex in the effective potential which is not strictly local. However, in the limit of low energy, this vertex reduces to

$$\sim \frac{1}{2M_0} \int d^4x d^2\theta \Big(\phi_1 \wedge \phi_2 \Big)^2 . \tag{4.28}$$

Eq. (4.28) is potentially interesting because it breaks the original "flavour" symmetry of SU(2) among $(\phi_i)_{i=1}^3$ to SO(2) (the rotations around ϕ_3). The effect of such a term is negligible in the limit $M_0 \to \infty$.

5. Conclusions and problems

In this note we have presented an elementary QFT computation of the glueball superpotential of the pure $\mathcal{N} = 1$ SYM model.

In spite of the simplicity of our method, we have obtained the standard Veneziano-Yankielowicz potential with minima of the "right" order of magnitude and correct multiplicity structure for the model we have studied.

As already noted in the introduction, the covariant super Feynman graph technique we have adopted is perturbatively equivalent to the matrix model approach [2, 3].

Only in the case that one applies directly the method of [3], UV divergences are likely to be run into, in particular at the one-loop level. In [7], one indeed obtains the leading VY potential, which, however, contains a large UV cutoff parameter.

On the other hand, if one attempts the same computation making use of the MM correspondence [2], one might expect to get the same result, but without encountering any UV divergences. Indeed, this appears to be the result of the recent piece of work, [8], where the leading term is again given by the VY potential, but the UV cutoff is replaced by physical finite parameters ($\sqrt[3]{m\Lambda^2_{\mathcal{N}=2}}$).

From [8], the general attitude is somewhat incomprehensible of limiting the application of the matrix model approach to the perturbative corrections to the VY potential [5].

Our method, based on the use of covariant super Feynman graphs [3], is less likely to be troubled by UV divergences, even in the computation of the leading term of the superpotential. This is due to the incorporation of the ERG method and the corresponding wilsonian action. As a matter of fact, our method must be considered as the closest possible QFT equivalent of the MM approach.

It is fairly straightforward to reproduce the result on the VY potential in [8] within our method, although the model there considered is different, being $\mathcal{N} = 1$ SYM with an additional chiral superfield in the adjoint representation.

In detail, in the calculation presented in section 4, one should ascribe the $\phi(= \phi_3)$ field with the independent physical mass m, instead of M_0, making it a physical degree of freedom. (This is analogous to the case of $\mathcal{N} = 4$ SYM regularization of the $\mathcal{N} = 2$ model [14].)

Repeating, step by step, the computations of section 4 with ϕ having physical mass m, and RG-reducing m to some other value $m' \ll m$, we obtain, instead of eq. (4.20),

$$\frac{i}{4}\frac{W^2}{16\pi^2}\ln\left(\frac{m' + \frac{S}{32\pi^2\cdot 3\cdot M^2}}{m + \frac{S}{32\pi^2\cdot 3\cdot M^2}}\right)^2 . \tag{5.1}$$

One may get rid of the kinematical term by choosing

$$\left(\ln\left(\frac{M}{M_0}\right)^2 + \frac{4\pi}{g_0^2}\right)_{M=\Lambda_{\mathcal{N}=2}} = 0 \qquad \text{for } G = \mathrm{SU}(2)\,, \tag{5.2}$$

that is defining $\Lambda_{\mathcal{N}=2}$ to be the value at which the inverse gauge coupling in the $\mathcal{N} = 2$ model vanishes.

Thus the effective potential is

$$\frac{i}{4}\frac{W^2}{16\pi^2}\ln\left(\frac{m' + \frac{S}{32\pi^2\cdot 3\cdot \Lambda_{\mathcal{N}=2}{}^2}}{m + \frac{S}{32\pi^2\cdot 3\cdot \Lambda_{\mathcal{N}=2}{}^2}}\right) . \tag{5.3}$$

By choosing $m' \ll m$ and assuming $\frac{S}{32\pi^2\cdot 3\cdot m\Lambda_{\mathcal{N}=2}{}^2} \ll 1$, the superpotential becomes proportional to

$$\left(2S\ln\left(\frac{S}{32\pi^2\cdot 3\cdot m\Lambda_{\mathcal{N}=2}{}^2}\right) + i\theta_0\right) , \tag{5.4}$$

which is the special case of the result in [8] for $N_c = 2$.

In order to obtain the glueball superpotential in $\mathcal{N} = 1$ SYM model, we have made rather heavy use of ERG techniques, proposed in [10, 14]

In the past, the VY term for $\mathcal{N} = 1$ SYM has been obtained as a one-loop effect in the context of the correspondence between $\mathcal{N} = 2$ supersymmetric sigma model in 2D and $\mathcal{N} = 1$ SYM in 4D, established through T_2 compactification. The superpotential is supposed to be immune to volume effects in the compactified space [9]. In our approach, instead of dimensional reduction, the ERG method produces convergent expressions directly in four dimensions.

JHEP05(2004)031

On the other hand, as noted in [10], our method is applicable only to those models which can be obtained as mass deformation of $\mathcal{N}=4$ SYM. A little wider class of models can be dealt with by mass deforming the $\mathcal{N}=2$ SQCD with vanishing one-loop beta function, i.e. with twice as many flavours as colours. Even then, though, one cannot study any general SGFT, whereas in the MM approach such restriction seems not to be present.

As it is, our method — which can be said to be the "QFT version" of MM techniques — does not reach as yet the same level of unified understanding of SUSY GFT the string approach does [26].

Note added. After the completion of the present work, the paper "Veneziano-Yankielovicz superpotential terms in $N=1$ SUSY gauge theories" by Gripaios and Wheater [29] has come to our notice.

In this work, the authors take a "current algebra" approach, i.e. they rely on the Konishi anomalous Ward-Takahashi identities (AWTI) [19] (see also [5]), instead of the Feynman graph approach of [7] and the present work.

Out of the well-known Konishi-type AWTI, the authors have succeeded in extracting the glueball superpotential (the part relative to the pure glueball dynamics) as the residue after the decoupling of the heavy fields, in the limit that the quark mass, m, and the Higgs-induced gauge boson mass, $\sqrt{m/2\lambda}$, become infinite. In this respect, this piece of work is not dissimilar to ours.

Since the approach in [29] depends only on the Konishi anomaly and the corresponding supercurrent divergence equation, rather than on the microscopic lagrangian (with its regularization problems), it has the advantage of allowing for the computation of quantities of interest for a wide class of gauge groups and hypermultiplet contents.

We believe this work supplies the important link for understanding glueball dynamics which has been mentioned in the introductory section of the present work.

We thank the referee for drawing our attention to the existence of this paper.

Acknowledgments

We would like to thank G. Immirzi, H. Itoyama, G. C. Rossi, S. Sugimoto and D. Zanon for helpful discussions. Part of this work has been carried out while K.Y. was a guest at YITP, Kyoto and at the Department of Physics, the University of Tokyo. It is a pleasure for him to thank K. Kugo, M. Ninomiya and K. Fujikawa for their warm hospitality.

A. The holomorphic and canonical coupling constants

As stated in section 3, throughout the paper we have been using the so-called holomorphic form for the action, eq. (3.1), and the corresponding gauge coupling constant,[8] eq. (3.2).

Its running, even in the $\mathcal{N}=1$ SYM model, is exausted at one loop, the corresponding β function being

$$\beta(g_h) = -\frac{3N_c}{16\pi^2}\, g_h^3\,.$$

[8]In this appendix its real part will be denoted by g_h.

JHEP05(2004)031

If the dinamically generated scale, $\Lambda_{\mathcal{N}=1}$, is defined as the value at which the inverse coupling constant vanishes, one gets precisely (cf. eq. (3.6))

$$\Lambda_{\mathcal{N}=1} = M_0 \exp -\frac{8\pi^2}{3N_c\, g_h^2}\,.$$

Going over to the more conventional canonical form is not completely trivial, and requires the use of the Konishi anomaly [19]. The relation between the two coupling constants at a given scale μ is [10]

$$\frac{1}{g_h^2(\mu)} = \frac{1}{g_c^2(\mu)} + \frac{N_c}{4\pi^2} \ln g_c(\mu)\,.$$

(g_c only runs down to $\mu_0 = 8\pi^2 \mathrm{e}\Lambda/N_c$ [27].)

Substituting the above in the expression of $\Lambda_{\mathcal{N}=1}$ gives

$$\Lambda_{\mathcal{N}=1} = \frac{M_0}{g_c^{2/3}(M_0)} \exp -\frac{8\pi^2}{3N_c\, g_c^2(M_0)}\,,$$

which has the correct dependence upon g_c as expected from explicit instanton calculations [28].

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[3] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, *Perturbative computation of glueball superpotentials*, *Phys. Lett.* **B 573** (2003) 138 [hep-th/0211017];
R. Dijkgraaf, M.T. Grisaru, H. Ooguri, C. Vafa and D. Zanon, *Planar gravitational corrections for supersymmetric gauge theories*, *J. High Energy Phys.* **04** (2004) 028 [hep-th/0310061].

[4] M.T. Grisaru and D. Zanon, *Covariant supergraphs, 1. Yang-Mills theory*, *Nucl. Phys.* **B 252** (1985) 578; *Covariant supergraphs, 2. Supergravity*, *Nucl. Phys.* **B 252** (1985) 591.

[5] R. Argurio, G. Ferretti and R. Heise, *An introduction to supersymmetric gauge theories and matrix models*, hep-th/0311066.

[6] G. Veneziano and S. Yankielowicz, *An effective lagrangian for the pure $N = 1$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 113** (1982) 231.

[7] J. Ambjørn and R.A. Janik, *Towards a diagrammatic derivation of the Veneziano-Yankielowicz-Taylor superpotential*, *Phys. Lett.* **B 569** (2003) 81 [hep-th/0306242].

[8] G. Hailu and H. Georgi, *On exact superpotentials, free energies and matrix models*, *J. High Energy Phys.* **02** (2004) 038 [hep-th/0401101].

[9] M. Aganagic, K. Intriligator, C. Vafa and N.P. Warner, *The glueball superpotential*, hep-th/0304271.

JHEP05(2004)031

[10] N. Arkani-Hamed and H. Murayama, *Holomorphy, rescaling anomalies and exact beta functions in supersymmetric gauge theories*, *J. High Energy Phys.* **06** (2000) 030 [hep-th/9707133].

[11] J. Polchinski, *Renormalization and effective lagrangians*, *Nucl. Phys.* **B 231** (1984) 269.

[12] G. Gallavotti, *Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods*, *Rev. Mod. Phys.* **57** (1985) 471.

[13] T.R. Morris, *The exact renormalization group and approximate solutions*, *Int. J. Mod. Phys.* **A 9** (1994) 2411 [hep-ph/9308265].

[14] S. Arnone, C. Fusi and K. Yoshida, *Exact renormalization group equation in presence of rescaling anomaly*, *J. High Energy Phys.* **02** (1999) 022 [hep-th/9812022].

[15] I.L. Buchbinder, *Progress in study of $N = 4$ SYM effective action*, hep-th/0402089;
I.L. Buchbinder and S.M. Kuzenko, *Comments on the background field method in harmonic superspace: non-holomorphic corrections in $N = 4$ SYM*, *Mod. Phys. Lett.* **A 13** (1998) 1623 [hep-th/9804168];
M. Dine and N. Seiberg, *Comments on higher derivative operators in some SUSY field theories*, *Phys. Lett.* **B 409** (1997) 239 [hep-th/9705057];
See also N. Dorey, *An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory*, *J. High Energy Phys.* **07** (1999) 021 [hep-th/9906011].

[16] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063].

[17] S. Kovacs, *A perturbative re-analysis of $N = 4$ supersymmetric Yang-Mills theory*, hep-th/9902047;
S. Kovacs, *$N = 4$ supersymmetric Yang-Mills theory and the AdS/SCFT correspondence*, hep-th/9908171.

[18] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1992.

[19] K.-I. Konishi and K.-I. Shizuya, *Functional integral approach to chiral anomalies in supersymmetric gauge theories*, *Nuovo Cim.* **A90** (1985) 111;
F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *J. High Energy Phys.* **12** (2002) 071 [hep-th/0211170].

[20] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200].

[21] M. D'Attanasio and T.R. Morris, *Gauge invariance, the quantum action principle and the renormalization group*, *Phys. Lett.* **B 378** (1996) 213 [hep-th/9602156];
T.R. Morris, *A gauge invariant exact renormalization group, I*, *Nucl. Phys.* **B 573** (2000) 97 [hep-th/9910058]; *A gauge invariant exact renormalization group, II*, *J. High Energy Phys.* **12** (2000) 012 [hep-th/0006064];
S. Arnone, A. Gatti and T.R. Morris, *A proposal for a manifestly gauge invariant and universal calculus in Yang-Mills theory*, *Phys. Rev.* **D 67** (2003) 085003 [hep-th/0209162].

[22] M.F. Sohnius and P.C. West, *Conformal invariance in $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 100** (1981) 245;
M.T. Grisaru and W. Siegel, *Supergraphity, 2. Manifestly covariant rules and higher loop finiteness*, *Nucl. Phys.* **B 201** (1982) 292.

[23] J. Zinn-Justin, *Quantum field theory and critical phenomena*, Clarendon Press, Oxford 1996, appendix 10, p. 221.

[24] M. Bonini, M. D'Attanasio and G. Marchesini, *Brs symmetry for Yang-Mills theory with exact renormalization group*, *Nucl. Phys.* **B 437** (1995) 163 [hep-th/9410138].

[25] S. Arnone, D. Francia and K. Yoshida, *Exact renormalization group equation in presence of rescaling anomaly, II. The local potential approximation*, *Mod. Phys. Lett.* **A 17** (2002) 1191 [hep-th/0110104].

[26] See e.g. the most recent reviews on the subject, E. Imeroni and A. Lerda, *Non-perturbative gauge superpotentials from supergravity*, *J. High Energy Phys.* **12** (2003) 051 [hep-th/0310157];
A. Liccardo, R. Marotta and F. Pezzella, *Anomalies and tadpoles in open/closed string duality*, *Class. and Quant. Grav.* **21** (2004) S1433 [hep-th/0312248].

[27] G. Carlino, K. Konishi, N. Maggiore and N. Magnoli, *On the beta function in supersymmetric gauge theories*, *Phys. Lett.* **B 455** (1999) 171 [hep-th/9902162].

[28] A. Armoni, M. Shifman and G. Veneziano, *Exact results in non-supersymmetric large-N orientifold field theories*, *Nucl. Phys.* **B 667** (2003) 170 [hep-th/0302163].

[29] B.M. Gripaios and J.F. Wheater, *Veneziano-Yankielowicz superpotential terms in $N = 1$ SUSY gauge theories*, *Phys. Lett.* **B 587** (2004) 150 [hep-th/0307176].

JHEP05(2004)031